Complex Numbers

Circular functions – Powers, multiple angles, series

1. Prove that if $2\cos\theta = x + \frac{1}{x}$, then $2\cos n\theta = x^n + \frac{1}{x^n}$.

Hence or otherwise solve the equation $5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0$.

2. If $u = \cos\frac{2}{7}\pi + i\sin\frac{2}{7}\pi$, state the value of u^7 and deduce the value of $1 + u + u^2 + u^3 + u^4 + u^5 + u^6$.

If $\alpha = u + u^2 + u^4$, $\beta = u^3 + u^5 + u^6$,

find $\alpha + \beta$ and $\alpha\beta$, and hence write down the quadratic equation whose roots are α and β .

Deduce that : $\cos\frac{2}{7}\pi + \cos\frac{4}{7}\pi + \cos\frac{8}{7}\pi = -\frac{1}{2}$ and $\sin\frac{2}{7}\pi + \sin\frac{4}{7}\pi + \sin\frac{8}{7}\pi = \frac{1}{2}\sqrt{7}$.

3. Show that :

(i) $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

(ii) $64\sin^7\theta = 35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta$

(iii) $2^6\sin^5\theta\cos^2\theta = \sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta$

(iv) $64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35$.

4. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, prove that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$$

and $$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$$

by substituting $p = \text{cis}\,\alpha$, $q = \text{cis}\,\beta$, $r = \text{cis}\,\gamma$, and using the factorization of $p^3 + q^3 + r^3 - 3pqr$.

5. If $2\cos x = u + \frac{1}{u}$, $2\cos y = v + \frac{1}{v}$, prove that $2\cos(mx + ny) = u^m v^n + \frac{1}{u^m v^n}$.

6. Find an expression for $\tan 9\theta$ in terms of $\tan\theta$.

By means of the equation $\tan 9\theta = 0$, show that $\tan\frac{r\pi}{9}$, where $r = 1, 2, 3, \dots, 8$,

are roots of the equation $9 - 84x^2 + 126x^4 - 36x^6 + x^8 = 0$,

and that $\tan^2\frac{r\pi}{9}$, where $r = 1, 2, 3, 4$ are the roots of the equation

$$9 - 84y + 126y^2 - 36y^3 + y^4 = 0 .$$

Hence show that $\tan^2\frac{\pi}{9} + \tan^2\frac{2\pi}{9} + \tan^2\frac{3\pi}{9} + \tan^2\frac{4\pi}{9} = 36$

and $\tan\frac{\pi}{9}\tan\frac{2\pi}{9}\tan\frac{3\pi}{9}\tan\frac{4\pi}{9} = 3$.

7. Let $\binom{n}{k}$ be the coefficient of x^k in the binomial expansion of $(1 + x)^n$.

By expanding $(1 + i)^{2n}$ or otherwise, prove that

(i) $\sum_{k=0}^{n}(-1)^k\binom{2n}{2k} = 2^n \cos\frac{n\pi}{2}$

(ii) $\sum_{k=0}^{n-1}(-1)^k\binom{2n}{2k+1} = 2^n \sin\frac{n\pi}{2}$.

8. The point representing the complex number z in an Argand diagram describes the circle $|z - 1| = 1$.
Show that $z = 1 + \cos\theta + i\sin\theta$, where $-\pi \le \theta \le \pi$,
and deduce that the point representing the complex number $1/z$ describes a straight line.

Find the modulus and argument of z and hence, or otherwise, express $(1 + \cos\theta + i\sin\theta)^n$ in the form $x + iy$, where n is a positive integer, and x and y are real.

By writing $\cos\theta + i\sin\theta = \omega$, using the binomial expansion of $(1 + \omega)^n$, prove that

$1 + C_1^n \cos\theta + C_2^n \cos 2\theta + ... + C_n^n \cos n\theta = \left(2\cos\frac{\theta}{2}\right)^n \cos\frac{n\theta}{2}$.

9. Show that $\sum_{r=1}^{n}\binom{n}{r}\sin 2r\theta = 2^n \cos^n\theta \sin n\theta$.

10. Sum each of the following series :

(i) $x\sin\theta + x^2 \sin 2\theta + x^3 \sin 3\theta + \ldots + x^{n-1}\sin(n-1)\theta$,

(ii) $\sum_{r=0}^{n-1}(r+1)\sin r\theta$

(iii) $\sum_{r=1}^{n}\sin^r\theta \sin r\theta$

11. Show that, if n is a positive integer,

$$(-1)^n 2^{2n} \sin^{2n+1}\theta$$
$$= \sin(2n+1)\theta - C_1^{2n+1}\sin(2n-1)\theta + ... + (-1)^r C_r^{2n+1}\sin(2n+1-2r)\theta + ... + (-1)^n C_n^{2n+1}\sin\theta \quad .$$

12. If n is odd , prove that $\sin n\theta$ can be expressed in the form

$$\sin n\theta = b_1 s + b_3 s^3 + \ldots + b_n s^n ,$$

where $s = \sin\theta$, and $b_1 , b_3 , \ldots, b_n$ are real numbers independent of θ .

Find b_1 and b_n .

Also prove that $b_{r+2} = \frac{r^2 - n^2}{(r+1)(r+2)} b_r$ and hence find b_3 .